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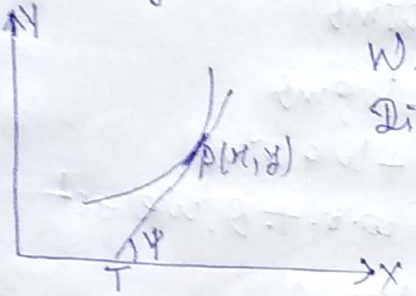
Differential Calculus (curvature)

Q.1. Find the radius of curvature for the Cartesian curve $y = f(x)$.

Soln:



We know that $\tan\psi = \dfrac{dy}{dx}$

Differentiating w.r.t. x, we get

$$\sec^2\psi \frac{d\psi}{dx} = \frac{d^2y}{dx^2}$$

$$\Rightarrow (1+\tan^2\psi)\frac{d\psi}{ds}\cdot\frac{ds}{dx} = \frac{d^2y}{dx^2}$$

$$\Rightarrow (1+\tan^2\psi)\frac{1}{\rho}\cdot\sec\psi = \frac{d^2y}{dx^2}$$

$$\Rightarrow (1+\tan^2\psi)\frac{1}{\rho}\left[\sqrt{1+\tan^2\psi}\right] = \frac{d^2y}{dx^2}$$

$$\Rightarrow \frac{1}{\rho}(1+\tan^2\psi)^{\frac{3}{2}} = \frac{d^2y}{dx^2}$$

$$\Rightarrow \frac{1}{\rho}\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}} = \frac{d^2y}{dx^2}$$

Hence, $\rho = \dfrac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$, provided $\dfrac{d^2y}{dx^2} \neq 0$.

$$= \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2} = \frac{(1+p^2)^{\frac{3}{2}}}{q}$$ solved

Q.2. Find the radius of curvature at any point of the curve $r = a(1+\cos\theta)$

Soln: Here, $r_1 = -a\sin\theta$, $r_2 = -a\cos\theta$

$$\therefore \rho = \frac{(r^2+r_1^2)^{\frac{3}{2}}}{r^2+2r_1^2-rr_2} = \frac{[a^2(1+\cos\theta)^2+a^2\sin^2\theta]^{\frac{3}{2}}}{a^2(1+\cos\theta)^2+2a^2\sin^2\theta+a^2\cos\theta(1+\cos\theta)}$$

$$= \frac{a^3(2+2\cos\theta)^{3/2}}{a^2[3+3\cos\theta]} = \frac{2^{\frac{3}{2}}}{3}a(1+\cos\theta)^{\frac{1}{2}} = \frac{2^{3/2}}{3}a^{1/2}\sqrt{r}$$

$$\therefore \rho = \frac{2^{\frac{3}{2}}}{3}a^{\frac{1}{2}}\sqrt{r} = (\text{constant})\sqrt{r} \quad \therefore \rho \propto \sqrt{r}$$

solved

Differential Calculus (Curvature)

Q.3. For the curve $r^m = a^m \cos m\theta$, prove that

$$\rho = \frac{a^m}{(m+1)r^{m-1}}$$

Soln: Now, $r^m = a^m \cos m\theta$

Taking logarithm of both sides, we get

$$\log r^m = \log(a^m \cos m\theta)$$

$$\Rightarrow m\log r = \log a^m + \log \cos m\theta$$

$$\Rightarrow m\log r = m\log a + \log \cos m\theta$$

Differentiating both sides w.r.t. θ, we get

$$m\cdot\frac{1}{r}\cdot\frac{dr}{d\theta} = -m\tan m\theta$$

$$\Rightarrow \cot\phi = \cot\left(\frac{\pi}{2}+m\theta\right)$$

$$\therefore \phi = \frac{\pi}{2}+m\theta$$

Again, $p = r\sin\phi = r\sin\left(\frac{\pi}{2}+m\theta\right) = r\cos m\theta$

$$\Rightarrow p = r\cdot\frac{r^m}{a^m} \Rightarrow a^m p = r^{m+1}$$

Differentiating both sides w.r.t. to r, we get

$$a^m\frac{dp}{dr} = (m+1)r^m$$

$$\therefore \rho = r\frac{dr}{dp} = r\cdot\frac{a^m}{(m+1)r^m} = \frac{a^m}{(m+1)r^{m-1}}$$ Proved.

Q.4. For any curve prove the formula

$$\rho = \frac{r}{\sin\phi\left(1+\frac{d\phi}{d\theta}\right)}, \text{ where } \tan\phi = r\frac{d\theta}{dr}.$$

Soln: We know that $\sin\phi = r\frac{d\theta}{ds}$, $\rho = \frac{ds}{d\psi}$ and $\theta+\phi=\psi$.

Now, $\sin\phi\left(1+\frac{d\phi}{d\theta}\right) = r\frac{d\theta}{ds} + r\frac{d\theta}{dr}\cdot\frac{d\phi}{d\theta} = \left(\frac{d\theta}{ds}+\frac{d\phi}{ds}\right)r$

$$= r\frac{d}{ds}(\theta+\phi) = r\frac{d\psi}{ds} = \frac{r}{\rho}$$

$$\therefore \rho = \frac{r}{\sin\phi\left(1+\frac{d\phi}{d\theta}\right)}$$

Proved.